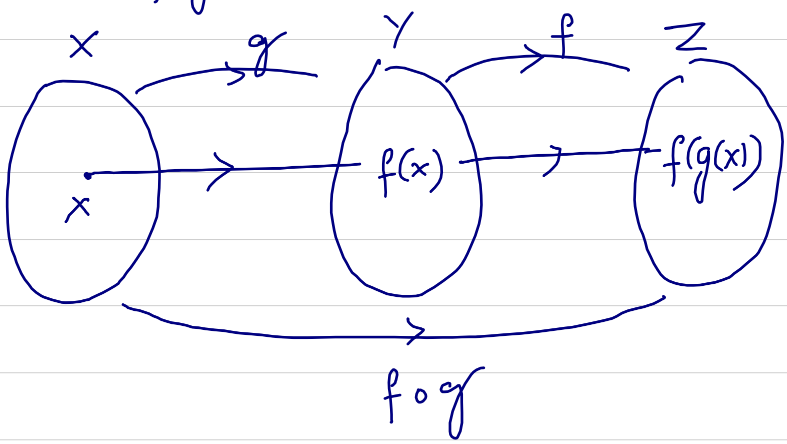
Composition of Functions

Let $g: X \to Y$ and $f: Y \to Z$ be functions. Then the composition of g and f is defined as the function $f \circ g : X \to Z$ such that $f\circ g(x) = f(g(x))$.



Attention : You are applying g first and then f. So the composition is $f \circ g$.

second func. first function

Exercise: Explain why.

The domain of $f\circ g$ is a subset of the domain of g. The range of $f \circ g$ is contained in the range of f.

Ex. $f(x) = x^2+1$ and $g(x) = x+2$

Then. $f\circ g(x) = f(g(x)) = f(x+2)$

$$= (x+2)^2+1$$
$$= x^2+4x+4+1$$
$$= x^2+4x+5$$

$$g\circ f(x) = g(f(x)) = g(x^2+1)$$
$$= x^2+1+2$$
$$= x^2+3$$

Ex. Given $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{1}{x}$, determine $f \circ g$, and its domain.

Soln. $f \circ g(x) = f(g(x))$

$$= f\left(\frac{1}{x}\right)$$

$$= \frac{1}{\frac{1}{x} - 1}$$

$$= \frac{1}{\frac{1-x}{x}}$$

$$= \frac{x}{1-x}$$

We know that domain of $f \circ g$ is contained in domain of g.

Domain of g: $g(x) = \frac{1}{x}$.

Hence Domain of $g = (-\infty, 0) \cup (0, \infty)$

Domain of $f \circ g$: There is $1-x$ is the denominator. It cannot be zero. When is it zero?

When $1 - x = 0$ or $x = 1$.

Thus, domain of $f \circ g$ does not include 1.

Since Domain of $f \circ g$ is contained in Domain of g. Domain of $f \circ g$ excludes 1 and 0.

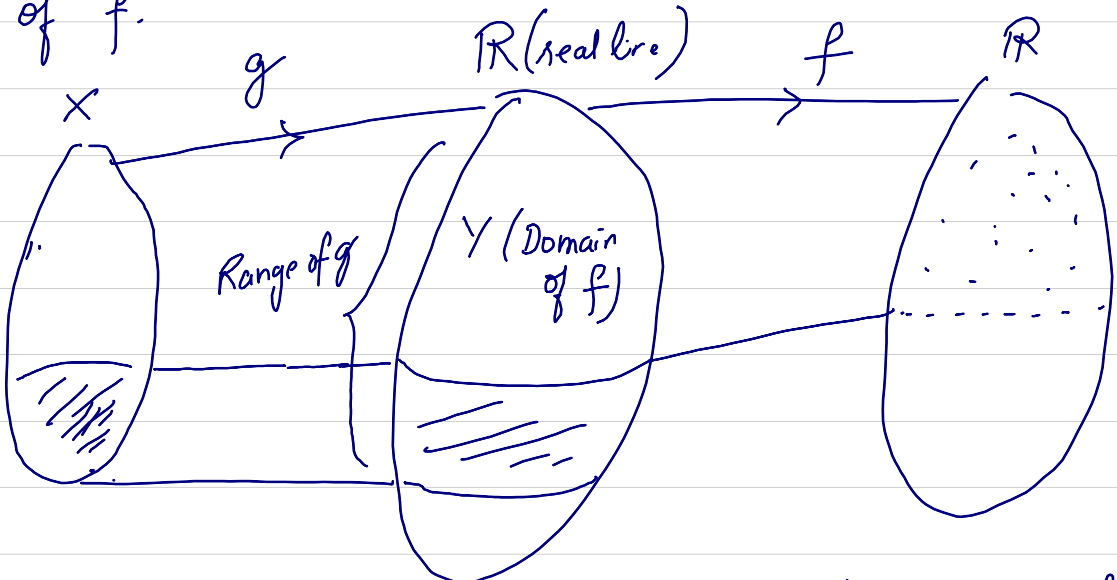
Thus, Domain of $f \circ g = (-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

ALTERNATIVE METHOD FOR FINDING DOMAIN OF COMPOSITES

It is possible to find the domain of a composite $f\circ g$ without finding the expression for $f\circ g$. This is useful as sometimes the expression for $f\circ g$ is messy.

Since we will deal only with numerical functions we may assume that the codomain for g is $\mathbb{R}$

Let X be the domain of g and Y be the domain of f.



You see how g can land outside the domain of f? In the diagram it is the shaded part. We cannot take the whole of domain of g as the domain of $g\circ f$ because after you apply g, you cannot apply f on the shaded part (f is only defined on Y)

So we must find all $x \in X$ such that $g(x)$ is contained in Y (domain of f).

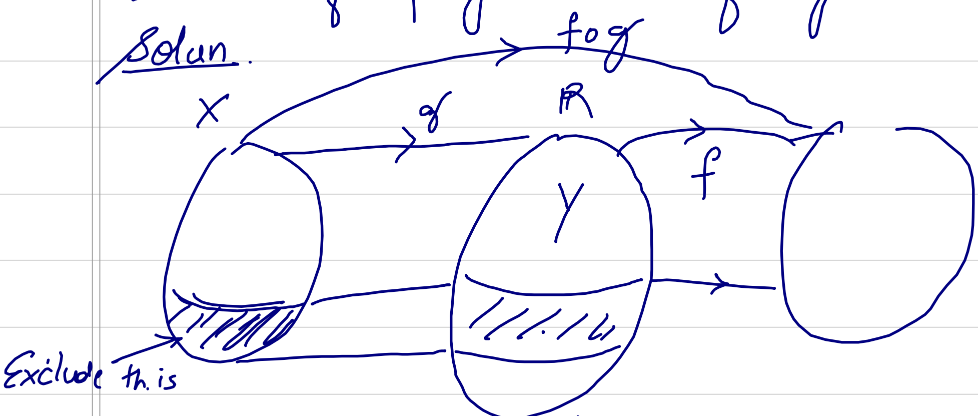
$\therefore$ Domain of $f\circ g = X -$ (shaded part of X)

Alternative method:

Determining domain of composites without finding the composite.

Let $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x+3}$. Find the domain of $f \circ g$ without finding the composite.

Soln.



First we need to find domain of g:

$$g(x) = \sqrt{x+3}.$$

Hence, $x+3 \geq 0$

or, $x \geq -3$

$\therefore$ Domain of $g = [-3, \infty)$ $\quad (X)$

Then, Domain of $f = (-\infty, 2) \cup (2, \infty)$. $\quad (Y)$

When does the range of g lie inside of domain of f?

Ans. When range of g does not include 2.

When does range of g not include 2?

Ans. Lets find when g has range 2:

Let $x \in X$ st. $g(x) = 2$

or, $\sqrt{x+3} = 2$

or, $x+3 = 4$

$\Rightarrow x = 1$

So Domain of $f \circ g$ is $[-3, 1) \cup (1, \infty)$.

Exercise Given $f(x) = x^3 - 3$
$g(x) = 1 + x^3$,
evaluate $f(g(1))$ and $g(f(1))$.